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CR 114708

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(NASA-CR-114708) CONCEPTS FOR A  
THEORETICAL AND EXPERIMENTAL STUDY OF  
LIFTING ROTOR RANDOM LOADS AND  
VIBRATIONS, PHASE 2 (Washington Univ.)  
56 p HC \$5.00

N74-14756

CSCI 01C

G3/02

Unclas  
26988

Concepts for a Theoretical and Experimental  
Study of Lifting Rotor Random Loads  
and Vibrations

Phase II Report under Contract NAS2-4151

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St. Louis, Missouri

August, 1968



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Prepared for the U. S. Army Aeronautical  
Research Laboratory at Ames Research Center,  
Moffett Field, California

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### Scope of Extended Contract NAS2-4151

Work under Contract NAS2-4151 started on February 1, 1967. Results obtained through August, 1967 are summarized in "Phase I Report under Contract NAS2-4151" of September, 1967. Subject contract was extended through July, 1968. The main purpose of the extended contract was to check the validity of the approximate digital method for computing the response of blade flapping to random inputs, tentatively suggested in Phase I Report, by comparison with NASA conducted simulator studies, to develop alternate methods if required and to extend the analysis to higher rotor advance ratios. This report summarizes the results obtained since September, 1967 through July, 1968, during which period 12.9 man-months were expended.

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Abstract:

A comparison with NASA conducted simulator studies has shown that the approximate digital method for computing rotor blade flapping responses to random inputs, tentatively suggested in Phase I Report, gives with increasing rotor advance ratio the wrong trend. Consequently, three alternative methods of solution have been considered and are described in this report. An approximate method based on the functional relation between input and output double frequency spectra, a numerical method based on the system responses to deterministic inputs and a perturbation approach. Among these the perturbation method requires the least amount of computation and has been developed in two forms - the first form to obtain the response correlation function and the second for the time averaged

spectra of flapping oscillations. The range of validity of the first form has been ascertained by a comparison between the Runge-Kutta and perturbation response values to harmonic inputs and that of the second form by comparing the time averaged response spectra values obtained from the perturbation method and the NASA conducted simulator results. Such comparisons indicate that the perturbation scheme should provide reasonable approximations up to a rotor advance ratio of one at a Lock blade inertia number of four.

Concepts for a Theoretical and Experimental  
Study of Lifting Rotor Random Loads  
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Notation

$\mu_x = E [x]$	Expected value of sample $x$
$E [f(x)]$	Expected value of sample function $f(x)$
$x_1 = x_1(t_1), x_2 = x_2(t_2)$	Values of sample function $x(t)$ at times $t_1$ and $t_2$ respectively
$t$	Time
$\tau = t_2 - t_1$	Time difference
$t = \frac{t_1 + t_2}{2}$	Average time
$R_{xy}(t_1, t_2)$	Cross-correlation function between sample time functions $x(t_1)$ and $y(t_2)$
$f = \frac{\omega}{2\pi}$	Frequency
$\Delta f$	Frequency interval
$X(f)$	Fourier transform of sample function $x(t)$
$S_{xy}(f_1, f_2) = E [X^*(f_1)Y(f_2)]$	Cross-correlation function between sample frequency functions $X^*(f_1)$ and $Y(f_2)$ , also called power spectral density
$h(\tau)$	Unit impulse response function
$H(f)$	Frequency response function

$F$	Modulating frequency
$\Omega_o$	Rotor angular velocity
$\mu$	Advance ratio
$\beta$	Blade flapping angle, positive up
$\bar{\alpha}$	Mean blade angle of attack
$S_{\bar{\alpha}}(f)$	Power spectral density of mean angle of attack
$\bar{S}_{\beta}(f)$	Average power spectral density of blade flapping angle
$A(t)$	Right hand side deterministic function
$y(\omega, t)$	Response of the system to the input $e^{i\omega t}$
$y_A(\omega, t)$	Response of the system to the input $A(t)e^{i\omega t}$
$y_c(\omega, t)$	Real part of $y(\omega, t)$
$y_s(\omega, t)$	Imaginary part of $y(\omega, t)$
$y_{Ac}(\omega, t)$	Real part of $y_A(\omega, t)$
$y_{As}(\omega, t)$	Imaginary part of $y(\omega, t)$

$c_1, c_2, d_1$  and  $d_2$

Constants associated with the  
left hand side of the blade  
flapping equation

$a_0, a_1, a_2, \dots, b_1, b_2, \dots$

Constants associated with  $A(t)$

$B(f)$

Fourier transform of  $\beta(t)$

$\overline{A}(f)$

Fourier transform of  $\overline{\alpha}(t)$

Superscripts:

\*

Conjugate complex

.

Time differentiation

—

Time average

## 1. Introduction

Of the various complications encountered when trying to apply to lifting rotors the stochastic methods developed to analyze airplane responses to atmospheric turbulence, we are concerned in this Phase II Report only with the time varying character of the system and of the random input. Though the theory is developed in a more general form applicable to the response of time varying linear systems to certain types of non-stationary random inputs, the application is to the flapping response of rigid blades hinged to a rigidly supported hub. The blades represent in forward flight an approximately linear system with time variable periodic stiffness and damping. Because of the periodically varying relative flow velocity occurring in forward flight of the lifting rotor, the aerodynamic excitation of the blades cannot be represented by a stationary random process as in the case of frozen wing aircraft, but must be described as a non-stationary stochastic input.

Non-stationary random inputs have been analyzed for a few engineering applications, for example for the response of airframes to random runway disturbances during decelerations after touchdown, Ref. (1), for the description of strong motion random earthquake excitation, Ref. (2), and for the response of spacecraft to time varying random excitations during the launching phase, Ref. (3). In these applications the system had constant parameters and could be represented by a time invariable transfer function. A flapping blade of a lifting rotor, however, because of the time variable periodic parameters, cannot be represented by such a time invariable transfer function.

The general theory of non-stationary stochastic processes has been well established as a direct extension of the corresponding theory of stationary random processes, References (4), (5), (6), (7). Rigorous solutions of responses to non-stationary random inputs thus far available are, however, restricted to constant parameter systems, References (6), (8). The complexity

of the analysis is due to the fact that, except in very special cases, closed form solutions valid over large time intervals do not as yet exist for differential equations with variable coefficients. When it is possible to find a rigorous solution, as in the case of the Bessel differential equation, the quadrature operations required to obtain for example the response auto-correlation function are quite involved even for stationary random inputs, Ref. (8).

Since our literature survey has not uncovered prior work toward solving the response of a linear system with time variable parameters under non-stationary stochastic loading, an approximate method for moderate advance ratio was suggested in Phase I Report according to which it was assumed that both the excitation and the response can be considered to be a stationary random process modulated by a deterministic time function. It was further assumed that the general equation between the two-frequency input and output power spectral densities could be approximately solved as far as the time averaged single frequency response spectrum is concerned, by ignoring the relations between the diagonal and off-diagonal terms of the two-frequency power spectra and by considering only the relation between the diagonal terms of the input and response spectra.

Since submitting Phase I Report consistent data have been received from the Simulator Computer Systems Branch of the NASA Ames Research Center, where the problem of random blade flapping response at various advance ratios had been simulated upon our request. When comparing the time averaged single frequency power spectra from the simulator study with the equivalent data from the approximate digital method suggested in Phase I Report, it was found that this method resulted with increasing rotor advance ratio in the wrong trend, so that the method cannot be accepted as an approximation. ©

Efforts were then directed toward solving the general equation between the two-frequency input and output power spectral densities including the relations between diagonal and off-diagonal terms. However, even for rather crude discretization of this functional equation one must obtain a computer solution for many hundred simultaneous linear equations between complex variables. It was found that a first computer program established for this purpose did not yield convergence of the iterations and this attempt was then suspended, though further work on the computer program might still lead to a success.

Next it was considered to obtain a solution based on the response to deterministic inputs assuming zero initial displacement and rate of displacement. Once such response time histories have been obtained for a sufficient number of frequency intervals, it is a simple matter of a frequency quadrature to obtain time variable response mean square values.

It was finally decided, before attempting an entirely numerical solution, to develop a perturbation method of solution which is an approximate analytical method for cases where the time varying parameters in the differential equations do not differ very much from their time mean. A numerical solution of the deterministic response problem with its inherent computer costs can then be avoided.

Much of the reliability of the perturbation method within its range of applicability stems from the fact that it repeatedly deals with constant parameter systems for different known inputs.

Numerical evaluations in this report are concerned with the determination of the range of validity of the perturbation method and its application to the problem of random rigid blade flapping.

## 2. General Relations and Concepts for Non-Stationary Random Processes

Non-stationary random processes can be represented either by a double frequency power spectrum or in some cases by an instantaneous time varying spectrum. In the first two sub-sections these two important concepts are briefly discussed. The third sub-section deals with a particular non-stationary random process obtained by modulation of a stationary process with a deterministic time function. In the subsequent section on random response analyses it will be assumed that the non-stationary input is of this form.

### 2.1 Double Frequency Spectra

The general relations between correlation functions and power spectral densities have been discussed in Phase I Report Section 2.1. For two non-stationary random processes with sample functions  $x(t)$ ,  $y(t)$  having zero means and having the sample Fourier transforms  $X(f)$ ,  $Y(f)$  the cross-correlation function is given by

$$R_{xy}(t_1, t_2) = E [x(t_1)y(t_2)] = \iint_{-\infty}^{\infty} S_{xy}(f_1, f_2) e^{-12\pi(f_1 t_1 - f_2 t_2)} df_1 df_2 \quad 2.1$$

and the cross-power spectral density is given by the inverse of eqn. 2.1

$$S_{xy}(f_1, f_2) = E [X^*(f_1)Y(f_2)] = \iint_{-\infty}^{\infty} R_{xy}(t_1, t_2) e^{12\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2 \quad 2.2$$

For a single non-stationary random process with sample function  $x(t)$ , autocorrelation function  $R_x(t_1, t_2)$  and power spectral density  $S_x(f_1, f_2)$  are related by

$$R_x(t_1, t_2) = E [x(t_1)x(t_2)] = \iint_{-\infty}^{\infty} S_x(f_1, f_2) e^{-12\pi(f_1 t_1 - f_2 t_2)} df_1 df_2 \quad 2.3$$

and its inverse

$$S_x(f_1, f_2) = E [X^*(f_1)X(f_2)] = \iint_{-\infty}^{\infty} R_x(t_1, t_2) e^{12\pi(f_1 t_1 - f_2 t_2)} dt_1 dt_2 \quad 2.4$$

The spectral function  $S_x(f_1, f_2)$  is in general for any combination of frequencies  $f_1, f_2$  a complex number and not physically realizable. For weakly stationary random processes

$$R_x(t_1, t_2) = R_x(t_2 - t_1) \quad 2.5$$

$$S_x(f_1, f_2) = S_x\left(\frac{f_1 + f_2}{2}\right) \delta(f_2 - f_1) \quad 2.6$$

where  $\delta(\dots)$  is the Dirac delta function with properties

$$\left. \begin{aligned} \delta(t) &= 0 & \text{if } t \neq 0 \\ \delta(t) &= \infty & \text{if } t = 0 \\ \int_{-\infty}^{\infty} \delta(t) dt &= \int_{\epsilon}^{\epsilon} \delta(t) dt = 1, & \epsilon > 0 \end{aligned} \right\} \quad 2.7$$

so that for a function  $\phi(t)$ :

$$\int_{-\infty}^{\infty} \delta(t - t_0) \phi(t) dt = \phi(t_0) \quad 2.8$$

Inserting 2.5 to 2.7 in eqn. 2.3 and 2.4 one obtains with  $t_2 - t_1 = \tau$  and  $f_1 = f_2 = f$

$$R_x(\tau) = E [x(t_1)x(t_2)] = \int_{-\infty}^{\infty} S_x(f) e^{12\pi f \tau} df \quad 2.9$$

$$S_x(f) = E [X^*(f)X(f)] = \int_{-\infty}^{\infty} R_x(\tau) e^{-12\pi f \tau} d\tau \quad 2.10$$

Since  $X^*(f)X(f)$  is real, the spectral function  $S_x(f)$  is also real and physically realizable.



## 2.2 Instantaneous and Time Averaged Spectra

In Phase I Report a specific example of a time averaged power spectrum was given in Section 2.3. Here the general concepts of instantaneous and time averaged power spectra will be discussed. In the definition of the double frequency power spectral density, eqn. 2.4, it is assumed that the sample functions  $x(t)$  are defined over an infinite time interval, though in actuality one has available only sample records observed over a finite duration. Furthermore, the double frequency spectrum is not a physically realizable quantity. The concept of an instantaneous power spectrum, Ref. (9), defined over a finite time interval and then time averaged, resulting in a physically realizable single frequency spectrum, allows to establish a relation between field observations and the theory of non-stationary random processes.

The instantaneous power spectrum  $S_x(f, t)$  is, according to Ref. (9), defined as

$$S_x(f, t) = \int_{-\infty}^{\infty} R_x(t - \frac{\tau}{2}, t + \frac{\tau}{2}) e^{-12\pi f \tau} d\tau \quad 2.11$$

It is also not physically realizable but it leads to a practical way of treating non-stationary random data by averaging over a sufficiently long time interval  $T$ . Denoting with  $\bar{R}_x(\tau)$  and  $\bar{S}_x(f)$  the time averaged autocorrelation function and power-spectral density respectively, one obtains from eqn. 2.11:

$$\bar{S}_x(f) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} S_x(f, t) dt = \int_{-\infty}^{\infty} \bar{R}_x(\tau) e^{-12\pi f \tau} d\tau \quad 2.12$$

where

$$\begin{aligned} \bar{R}_x(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} R_x(t - \frac{\tau}{2}, t + \frac{\tau}{2}) dt \\ &= \int_{-\infty}^{\infty} \bar{S}_x(f) e^{12\pi f \tau} df \end{aligned} \quad 2.13$$

In order to relate the double frequency spectrum  $S_x(f_1, f_2)$  to the time averaged spectrum  $\bar{S}_x(f)$  we substitute in eqn. 2.3  $t_1 = t - \frac{T}{2}$ ,  $t_2 = t + \frac{T}{2}$ , so that

$$R_x(t - \frac{T}{2}, t + \frac{T}{2}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_x(f_1, f_2) e^{i2\pi t(f_2 - f_1)} e^{i\pi T(f_2 + f_1)} df_1 df_2$$

Therefore from eqn. 2.13

$$\begin{aligned} \bar{R}_x(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_x(f_1, f_2) e^{i2\pi t(f_2 - f_1)} e^{i\pi T(f_2 + f_1)} df_1 df_2 dt \\ &= \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_x(f_1, f_2) e^{i\pi T(f_2 + f_1)} \frac{1}{T} \int_{-T/2}^{T/2} e^{i2\pi t(f_2 - f_1)} dt df_1 df_2 \\ &= \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_x(f_1, f_2) e^{i\pi T(f_2 + f_1)} \frac{\sin(f_2 - f_1)\pi T}{(f_2 - f_1)\pi T} df_1 df_2 \end{aligned}$$

2.14

In accordance with Ref. (7), p. 450 it is now assumed that the double frequency spectrum of the non stationary random process can be expressed as a sum of regular and singular masses:

$$S_x(f_1, f_2) = S_r(f_1, f_2) + S_s(f_1) \delta(f_2 - f_1)$$

where  $S_r(f_1, f_2)$  has no line masses on the line  $f_1 = f_2$ . Inserting this expression into eqn. 2.14 and noting that

$$\lim_{T \rightarrow \infty} \frac{\sin(f_2 - f_1)\pi T}{(f_2 - f_1)\pi T} = \begin{cases} 1 & \text{for } f_1 = f_2 \\ 0 & \text{for } f_1 \neq f_2 \end{cases}$$

one obtains with:  $\frac{f_1 + f_2}{2} = f$

$$\bar{R}_x(\tau) = \int_{-\infty}^{\infty} S_s(f) e^{i2\pi \tau f} df$$

and by comparison with eqn. 2.13

$$\bar{S}_x(f) = S_s(f) \quad 2.15$$

This important theorem says that the time averaged autocorrelation function  $\bar{R}_x(\tau)$  and the time averaged power-spectral density  $\bar{S}_x(f)$  are uniquely defined by the line masses of the double frequency spectrum along the diagonal  $f_1 = f_2$ . If these line masses are zero, then  $\bar{S}_x(f) = \bar{R}_x(\tau) = 0$ . In the following section it is shown that such line masses occur when the non-stationary process can be represented by a stationary random process modulated by a periodic time function.

### 2.3 Stationary Random Processes Modulated by a Periodic Time Function

Consider a sample function from a non-stationary random process

$$z(t) = A(t)x(t) \quad 2.16$$

where  $A(t)$  is a deterministic periodic time function and  $x(t)$  a sample function from a stationary random process.

We write the Fourier series for  $A(t)$  with the basic frequency  $f_0$  in the form

$$A(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik2\pi f_0 t} \quad 2.17$$

$A(t)$  must be real since it represents a physical quantity, so that  $c_k = c_{-k}^*$  and we can write eqn. 2.17 also in the form

$$A(t) = \sum_{k=-\infty}^{\infty} c_k^* e^{-ik2\pi f_0 t}$$

Applying the first part of eqn. 2.3, one obtains for the autocorrelation function

$$R_z(t_1, t_2) = R_x(t_1, t_2) \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_k^* c_l e^{i2\pi f_0 (-kt_1 + lt_2)} \quad 2.18$$

Introducing as before  $t_1 = t - \tau/2$ ,  $t_2 = t + \tau/2$  and considering that  $R_x(t, \tau)$  is independent of  $t$ , since  $x(t)$  is the sample function

of a stationary random process, eqn. 2.18 assumes the form

$$R_z(\tau, t) = R_x(\tau) \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_k^* c_l e^{12\pi f_0 (1-k)t + (1+k)\frac{\tau}{2}} \quad 2.19$$

Time averaging the autocorrelation function according to eqn. 2.13 and considering

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} e^{12\pi f_0 (1-k)t} dt = \begin{cases} 1 & \text{for } l = k \\ 0 & \text{for } l \neq k \end{cases}$$

one obtains

$$\bar{R}_z(\tau) = R_x(\tau) \sum_{k=-\infty}^{\infty} c_k^* c_k e^{12\pi f_0 k \tau} \quad 2.20$$

From the first part of eqn. 2.4 one obtains for the power spectral density

$$S_z(f_1, f_2) = \sum_{j=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} c_j^* c_p S_x\left(\frac{f_1 + f_2 - f_0(j+p)}{2}\right) \delta(f_2 - f_1 - f_0(-j+p)) \quad 2.21$$

The time averaged spectral density, according to the theorem proven in section 2.2, is equal to the line mass of the double frequency spectrum along the diagonal  $f_1 = f_2$ , so that with  $j = p$  and

$$\frac{f_1 + f_2}{2} = f$$

$$\bar{S}_z(f) = \sum_{j=-\infty}^{\infty} c_j^* c_j S_x(f - j f_0) \quad 2.22$$

The same result can also be derived by inserting eqn. 2.20 into eqn. 2.12.

### 3. Response of Time-Varying Linear Systems

In Section 2.4 of Phase I Report the response autocorrelation function and the response power spectral density were given in terms of the time variable impulse response function and of the time variable frequency response function. It was noted that numerical solutions would be very difficult since in general neither the impulse response function nor the frequency response function are given analytically.

Following some remarks by Sveshnikov in Ref. (8), p. 135 the problem is here reformulated by introducing the response  $y(f,t)$  of the linear system to an excitation  $e^{i2\pi ft}$  where the system is assumed to be at rest in its equilibrium position at the origin of time. It is shown how input-output relations between correlation and spectral functions can be expressed in terms of the particular solution  $y(f,t)$ .

#### 3.1 General Non-Stationary Random Input

The response  $y(t)$  of a time varying linear system with an infinite operating time is given by

$$y(t) = \int_{-\infty}^{\infty} h(\tau, t) x(t - \tau) d\tau \quad 3.1$$

Physical realizability requires that the impulse response function

$$h(\tau, t) = 0 \quad \text{for } \tau < 0.$$

Stability of the system requires that

$$\int_{-\infty}^{\infty} |h(\tau, t)| d\tau < \infty.$$

The input  $x(t)$  applied to the system at time  $t - \tau$  can be either a deterministic function or a sample function of a stochastic process.

Assuming a harmonic input

$$x(t) = e^{i2\pi ft} \quad 3.2$$

the output, according to eqn. 3.1 is

$$y(f,t) = \int_{-\infty}^{\infty} h(\tau,t) e^{12\pi f(t-\tau)} d\tau \quad 3.3$$

which can also be written as

$$y(f,t) = H(f,t) e^{12\pi f t} \quad 3.4$$

with  $H(f,t) = \int_{-\infty}^{\infty} h(\tau,t) e^{-12\pi f \tau} d\tau \quad 3.5$

For a general input  $x(t)$  we write the Fourier transform inverse

$$x(t) = \int_{-\infty}^{\infty} e^{12\pi f t} X(f) df \quad 3.6$$

Since the response to the input  $e^{12\pi f t}$  is  $y(f,t)$ , see equation 3.2 and 3.3, one can write the response to the input  $x(t)$  in the form

$$y(t) = \int_{-\infty}^{\infty} y(f,t) X(f) df \quad 3.7$$

$x(t)$  and  $y(t)$  are physical quantities and, therefore, real. From eqn. 3.6 it then follows that  $X(f) = X^*(-f)$  and from eqn. 3.7 that  $y(f,t) = y^*(-f,t)$ . The latter equation can now also be written as

$$y(t) = \int_{-\infty}^{\infty} y^*(f,t) X^*(f) df$$

Defining correlation functions and power spectral densities according to eqns. 2.1 to 2.4 and inserting eqns. 3.6 and 3.7 one obtains:

$$R_y(t_1, t_2) = \iint_{-\infty}^{\infty} y^*(f_3, t_1) y(f_4, t_2) S_x(f_3, f_4) df_3 df_4 \quad 3.8$$

$$R_{xy}(t_1, t_2) = \iint_{-\infty}^{\infty} y(f_4, t_2) e^{-12\pi f_3 t_1} S_x(f_3, f_4) df_3 df_4 \quad 3.9$$

$$S_y(f_1, f_2) = \iint_{-\infty}^{\infty} S_x(f_3, f_4) \left\{ \int_{-\infty}^{\infty} e^{12\pi f_1 t_1} y^*(f_3, t_1) dt_1 \right\} \left\{ \int_{-\infty}^{\infty} e^{-12\pi f_2 t_2} y(f_4, t_2) dt_2 \right\} df_3 df_4$$

3.10

$$S_{xy}(f_1, f_2) = \int_{-\infty}^{\infty} S_x(f_1, f_4) \left\{ \int_{-\infty}^{\infty} e^{-12\pi f_2 t} y(f_4, t) dt \right\} df_4$$

3.11

The time dependent mean square response, which gives in many applications adequate information, is obtained by setting in eqn. 3.8  $t_1 = t_2 = t$ . Thus

$$\sigma_y^2(t) = R_y(t, t) = \iint_{-\infty}^{\infty} y^*(f_3, t) y(f_4, t) S_x(f_3, f_4) df_3 df_4$$

3.12

If the input is a stationary random process the double integrals of the frequencies reduce to single integrals and one obtains for example for the response autocorrelation function

$$R_y(t_1, t_2) = \int_{-\infty}^{\infty} y^*(f, t_1) y(f, t_2) S_x(f) df$$

3.8a

and for the time variable mean square

$$\sigma_y^2(t) = R_y(t, t) = \int_{-\infty}^{\infty} y^*(f, t) y(f, t) S_x(f) df$$

3.12a

For constant parameter systems the frequency response function  $H(f)$  is independent of time, so that

$$y(f, t) = H(f) e^{12\pi f t}$$

3.13

The mean square response is then given by

$$\sigma_y^2(t) = R_y(t, t) = \iint_{-\infty}^{\infty} H^*(f_3) H(f_4) e^{-12\pi t(f_3 - f_4)} S_x(f_3, f_4) df_3 df_4 \quad 3.14$$

and is time variable. For stationary random input this reduces to the time invariable mean square response

$$\sigma_y^2 = R_y(0) = \int_{-\infty}^{\infty} H^*(f) H(f) S_x(f) df \quad 3.15$$

For time variable systems the particular solution  $y(f, t)$  has to be usually evaluated numerically. It is, therefore, preferable to avoid complex arithmetic in the numerical algorithm. Since the system is linear, the response can be expressed as

$$y(f, t) = y_c(f, t) + iy_s(f, t) \quad 3.16$$

where  $y_c(f, t)$  and  $y_s(f, t)$  are the responses to the inputs  $\cos 2\pi ft$  and  $\sin 2\pi ft$  respectively. By expressing the Fourier transform of a sample function by

$$X(f) = X_1(f) + iX_2(f)$$

and then applying the definition 2.4

one can easily show that the power spectral density can be expressed by

$$S_x(f_1, f_2) = S_{xR}(f_1, f_2) + iS_{xI}(f_1, f_2) \quad 3.17$$

with properties

$$S_{xR}(f_1, f_2) = S_{xR}(-f_1, -f_2) \quad 3.18$$

$$S_{xI}(f_1, f_2) = -S_{xI}(-f_1, -f_2) \quad 3.19$$



$$\begin{aligned}
 R_y(t_1, t_2) = & \iint_{-\infty}^{\infty} \left\{ y_c(f_1, t_1) y_c(f_2, t_2) \right. \\
 & + y_s(f_1, t_1) y_s(f_2, t_2) \left. \right\} S_{xR}(f_1, f_2) df_1 df_2 \\
 & + \iint_{-\infty}^{\infty} \left\{ y_s(f_1, t_1) y_c(f_2, t_2) \right. \\
 & - y_c(f_1, t_1) y_s(f_2, t_2) \left. \right\} S_{xI}(f_1, f_2) df_1 df_2
 \end{aligned}
 \tag{3.20}$$

$$\begin{aligned}
 R_{xy}(t_1, t_2) = & \iint_{-\infty}^{\infty} \left\{ y_c(f_2, t_2) \cos 2\pi f_1 t_1 \right. \\
 & + y_s(f_2, t_2) \sin 2\pi f_1 t_1 \left. \right\} S_{xR}(f_1, f_2) df_1 df_2 \\
 & + \iint_{-\infty}^{\infty} \left\{ y_c(f_2, t_2) \sin 2\pi f_1 t_1 \right. \\
 & - y_s(f_2, t_2) \cos 2\pi f_1 t_1 \left. \right\} S_{xI}(f_1, f_2) df_1 df_2
 \end{aligned}
 \tag{3.21}$$

For stationary random input equation 3.20 reduces to the single integral

$$R_y(t_1, t_2) = \int_{-\infty}^{\infty} \left\{ y_c(f, t_1) y_c(f, t_2) + y_s(f, t_1) y_s(f, t_2) \right\} S_x(f) df
 \tag{3.22}$$

The time variable mean square is then given by

$$\sigma_y^2(t) = R_y(t, t) = \int_{-\infty}^{\infty} \left\{ y_c^2(f, t) + y_s^2(f, t) \right\} S_x(f) df
 \tag{3.23}$$

an expression derived by Sveshnikov in Ref. (8), p. 136.

### 3.2 Modulated Stationary Random Input

It is now assumed that the input can be represented by eqn. 2.16 and 2.17 so that the input is a stationary random process modulated by a periodic time function. In section 2.3 it had been shown that such a random process leads to a special kind of double frequency power spectrum with line masses along the diagonal  $f_1 = f_2$ , see eqn. 2.21, so that an average power spectral density and an average autocorrelation function different from zero exist. Substituting the input spectrum eqn. 2.21 into eqn. 3.8 for the response autocorrelation function, one obtains

$$R_y(t_1, t_2) = \iint_{-\infty}^{\infty} y^*(f_1, t_1) y(f_2, t_2) \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_k^* c_l S_x \left( \frac{f_1 + f_2 - f_0(k+l)}{2} \right) \delta(f_2 - f_1 - f_0(-k+l)) df_1 df_2$$

By virtue of relation 2.8 the above expression reduces to

$$R_y(t_1, t_2) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_k^* c_l \int_{-\infty}^{\infty} y^*(f_1, t_1) y\{f_1 + f_0(1-k)\} S_x(f_1 - f_0 k) df_1$$

With the substitution  $f_1 - kf_0 = f$  the expression further simplifies to

$$R_y(t_1, t_2) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_k^* c_l \int_{-\infty}^{\infty} y^*\{(f + kf_0), t_1\} y\{(f + lf_0), t_2\} S_x(f) df$$

3.24

From eqn. 3.4:

$$\sum_{l=-\infty}^{\infty} c_l y(f + lf_0, t_2) = \sum_{l=-\infty}^{\infty} c_l H(f + lf_0, t_2) e^{12\pi(f + lf_0)t_2}$$

Because of eqn. 2.17 the left hand side of this equation is the response to the input

$$A(t)e^{12\pi ft} \quad 3.25$$

Denoting this response by  $Y_A(f,t)$ , one obtains finally

$$R_Y(t_1, t_2) = \int_{-\infty}^{\infty} Y_A^*(f, t_1) Y_A(f, t_2) S_X(f) df \quad 3.26$$

This equation has the same form as eqn. 3.8a for stationary random input, the only difference being that in case of a modulated stationary random input the response  $Y_A(f,t)$  to  $A(t)e^{12\pi ft}$  is to be used instead of the response  $y(f,t)$  to  $e^{12\pi ft}$ .

The time variable mean square is now

$$\sigma_Y^2(t) = R_Y(t, t) = \int_{-\infty}^{\infty} Y_A^*(f, t) Y_A(f, t) S_X(f) df \quad 3.27$$

and corresponds to eqn. 3.12a.

In real form equations 3.22 and 3.23 can be used, whereby merely  $y_c(f,t)$  and  $y_s(f,t)$  must be replaced by the responses  $Y_{cA}(f,t)$ ;  $Y_{sA}(f,t)$  to the inputs  $A(t) \cos 2\pi ft$  and  $A(t) \sin 2\pi ft$  respectively. From a computational point of view it is of importance that a non-stationary random input of the type considered here can be treated in the same way as a stationary random input.

#### 4. Methods of Approximate Solutions

In Phase I Report the blade angle of attack taken as an average over the blade span, had been assumed to represent a stationary stochastic process, and the aerodynamic loads on the flapping blade had been determined by modulating this stationary process with a periodic time function. In a more sophisticated theory the blade angle of attack will appear already as a modulation of a stationary stochastic process which has to be defined by the atmospheric turbulence penetrated by the aircraft at constant flight speed. From a measurement point of view it is almost a necessity to assume the input to be a modulated stationary stochastic process, whereby the underlying stationary process can be measured with respect to its power-spectral density or correlation functions. In contrast, the power-spectral densities of non-stationary processes cannot be measured in principle and the measurement of their correlation functions requires a large set of sample functions, usually not available. The basic assumption with respect to the stochastic structure of the input made in this report is for lifting rotors from a physical point of view plausible, from a measurement point of view almost required, and from a mathematical or computational point of view, as shown in the preceding sections, a very great simplification. The three approximate methods discussed in the following are all based on this particular assumption for the stochastic input. The presentation is further limited to a single degree of freedom linear system. For the actual lifting rotor more than one degree of freedom should be considered, whereby input and response would appear in matrix form and where the problem of cross correlation functions between the various degrees of freedom would occur, as discussed in Phase I Report. Finally, the actual lifting rotor description includes non-linearities which if small could be considered in the perturbation theory, but which would render the general theory presented herein inapplicable.

#### 4.1 Solution Based on Functional Relation Between Input and Output Double Frequency Spectra

To illustrate the method, we will consider the approximate differential equation for blade flapping, derived in Phase I Report\*\*

$$\ddot{\beta} + (c_1 + a_1 \sin t)\dot{\beta} + (c_2 + a_2 \cos t)\beta = (c_3 + a_3 \sin t)\bar{\alpha} \quad 4.1$$

Here  $\bar{\alpha}$ , the blade angle of attack averaged over the blade span, is assumed to represent a stationary random process. The right hand side of eqn. 4.1 is then a modulated stationary process and represents the input to the blade flapping equation. In general the factors for  $\dot{\beta}$  (damping of the flapping motion), for  $\beta$  (stiffness) and for  $\bar{\alpha}$  can also contain other terms of the respective truncated Fourier series.

Taking the Fourier transform on both sides of eqn. 4.1 and denoting the Fourier transform of  $\beta$  by  $B$ , that of  $\bar{\alpha}$  by  $\bar{A}$ , one obtains, as shown in Phase I Report

$$\begin{aligned} & -(2\pi f_2)^2 B(f_2) + c_1 2\pi i f_2 B(f_1) + a_1 \pi \left\{ \left(f_2 - \frac{1}{2\pi}\right) B\left(f_2 - \frac{1}{2\pi}\right) \right. \\ & \left. - \left(f_2 + \frac{1}{2\pi}\right) B\left(f_2 + \frac{1}{2\pi}\right) \right\} + c_2 B(f_2) + \frac{a_2}{2} \left\{ B\left(f_2 - \frac{1}{2\pi}\right) + B\left(f_2 + \frac{1}{2\pi}\right) \right\} = \\ & c_3 \bar{A}(f_2) + \frac{a_3}{2} \left\{ \bar{A}\left(f_2 - \frac{1}{2\pi}\right) + \bar{A}\left(f_2 + \frac{1}{2\pi}\right) \right\} \end{aligned} \quad 4.2$$

Taking the conjugate complex of this equation and substituting  $f_1$  for  $f_2$ , multiplying the two equations and taking the mathematical expectation of this product leads to a lengthy functional equation between the known power-spectral density for the angle of attack  $\bar{\alpha}$

$$E \left[ \bar{A}^*(f_1) \bar{A}(f_2) \right] = \begin{cases} 0 & \text{for } f_1 \neq f_2 \\ S_{\bar{\alpha}}(f) & \text{for } f_1 = f_2 = f \end{cases}$$

and the power spectral density for the flapping angle

$$E \left[ B^*(f_1) B(f_2) \right] = S_{\beta}(f_1, f_2)$$

\*\*In later sections constants  $a_1$  and  $a_2$  are replaced by  $d_1$  and  $d_2$  respectively and the input modulating function  $c_3 + a_3 \sin t$  is replaced by  $\frac{a_0}{2} + b \sin t$ .

For the particular case of eqn. 4.2 the response power spectrum  $S_{\beta}(f_1, f_2)$  consists only of line masses along the lines  $f_1 = f_2$  and  $f_1 - f_2 = \pm \frac{1}{2\pi}$ . If higher order terms of the truncated Fourier series in the coefficients of  $\dot{\beta}$ ,  $\beta$  and  $\bar{\alpha}$  in eqn. 4.2 are considered, the response spectrum  $S_{\beta}(f_1, f_2)$  contains also line masses along the lines  $f_1 - f_2 = \pm \frac{n}{2\pi}$ , where  $n = 1, 2, \dots$  up to the highest order of the truncated Fourier series.

Replacing  $f = k\Delta f$ , where  $\Delta f$  is a small frequency interval and  $k$  a discrete variable with values  $k = 1, 2, \dots, m$ , the functional equation for  $S_{\beta}(f_1, f_2)$  can be replaced by a system of complex linear equations. Since from physical considerations

$$S_{\beta}(k\Delta f, l\Delta f) \rightarrow 0$$

for sufficiently large  $k$  and  $l$ , one obtains a finite number of equations to compute the values  $S_{\beta}(k\Delta f, l\Delta f)$  for all discrete values of  $k, l$ , for which  $S_{\beta}$  is assumed to be different from zero.

The problem is now reduced to the inversion of a large number (in the order of many hundred) of linear equations for complex variables, having complex coefficients. In Phase I Report the relations between the line masses on the diagonal,  $f_1 = f_2$  and the line masses on the other two lines,  $f_1 - f_2 = \pm \frac{1}{2\pi}$  had been neglected, and only a relation between the line masses on the diagonal line retained. This leads to a real system of equations with real unknowns. Since submitting Phase I Report it was found that this incomplete system of equations does not yield approximation, since not even the trend with increasing magnitude of the periodic coefficient  $a_1, a_2, a_3$  in eqn. 4.1 is properly established by the incomplete system of equations.

It was subsequently attempted to solve the complete system of equations for the complex unknowns, however for the numerical input data considered (see Section 5), the numerical experience showed that the complex coefficient matrix is ill conditioned and not directly suited for standard iterative techniques. The program so far completed stores only non zero elements

and for any preassigned value of  $m$  generates its own complex coefficient matrix which is by a standard subroutine split into real arithmetic. This representation of a complex matrix in real form takes practically twice the original core storage but was found to be computationally more convenient and easily amenable for double precision. It seems necessary to generate a preconditioning matrix, by a trial and error procedure.

This aspect of the problem, using preconditioning matrices, has not yet been exploited. It is unlikely that the preconditioning operation and the subsequent inversion of the large matrix will be possible without a considerable amount of computer time per case. Of physical significance and subject to direct measurement is only that portion of the double frequency spectrum  $S_{\beta}(f_1, f_2)$  consisting of a line mass along the diagonal  $f_1 = f_2$ , since, according to Section 2.2, only the diagonal line mass contributes to the time averaged power-spectral density. However, contrary to the assumption made in Phase I, the relations involving the line masses on the non-diagonal lines cannot be neglected in the computation.

#### 4.2 Solution Based on the System Response to Deterministic Inputs

The response autocorrelation function  $R_y(t_1, t_2)$  can be either computed from eqn. 3.24, using the response  $y(f, t)$  to the input  $e^{i2\pi ft}$ , or it can be computed from eqn. 3.26, using the response  $Y_A(f, t)$  to the input  $A(t)e^{i2\pi ft}$ , the actual computations to be performed with the equivalent real form equations. The latter approach takes less machine time but does not economically permit a parametric variation in  $A(t)$ . For the first approach the response calculations are independent of the input modulating function  $A(t)$  and variations in this function are reflected merely in the double summation of eqn. 3.24.

In either case a deterministic response analysis over a sufficiently wide frequency and time range is required. Once the responses are determined, the autocorrelation function  $R_y(t_1, t_2)$  or the mean square response  $R_y(t, t)$  requires merely a single integration over the applicable frequency range.

In performing the numerical response computations the computer time involved should be an important factor in selecting a suitable one-step or a multi-step method. Truncation and numerical instability problems should not affect the reliability of the computations over sufficiently large time intervals. It is presumed that the round off errors can be checked with a double precision arithmetic - a provision easily available in present day computers.

For second order differential equations as in our blade flapping problem, it is possible by a standard substitution (Ref. 12, p. 227) to obtain another equation of the same order but without the first derivative terms. A multi-step method known as Noumerov's method (Ref. 13, p. 137 and Ref. 14, p. 301) was used for some sample runs to compute the blade flapping response for an input  $\cos 2\pi ft$ , beginning with zero displacement and rate of displacement for  $t = 0$ . This method has no stability problems and the truncation error is of the order of  $O(h^6)$ , Ref. 14, p. 301. In addition to the known zero displacement and displacement rate conditions the method requires one extra



starting value of the displacement which was computed from Taylor series. On an overall basis this multi-step method took more machine time than the one-step method described below.

This one-step method was the Runge-Kutta method of fourth order specially suited to second order differential equations, giving a truncation error of order  $O(h^5)$ . The program is based on the algorithm given in Ref. 12, page 238. Being a one-step method it is self starting without any stability problems. Numerical comparisons have been made with reduced step size and in some other cases with the perturbation theory. This rather heuristic approach toward truncation and round off problems indicates that the computational errors are too insignificant to affect the reliability of the response calculations.

After numerically computing the deterministic responses over an adequate range of frequency and time the quadrature operations based on the Simpson rule is carried out in accordance with equation 3.24 or 3.26.

Numerical experience thus far gained seems to indicate that with computer time of 40-45 minutes on machines comparable to the IBM 360-50 Model, a reasonably accurate time dependent mean square response can be obtained. This assumes that the periodically varying damping and stiffness functions are explicitly given in the form of Fourier series. Otherwise a separate subroutine has to be added to perform such a Fourier analysis.

#### 4.3 Perturbation Method for Linear Time Variable Systems with Small Time Varying Parameters

A drawback in a complete numerical approach is the considerable amount of machine time in computing the responses for inputs  $\cos \omega t$  and  $\sin \omega t$  over an adequate range of the discrete frequency parameter  $\omega$ . Approximate analytical methods on the other hand provide solutions in terms of the variable frequency  $\omega$  and it is possible to make a qualitative study of the response with or without including transient effects.

To evaluate the correlation function of the response by the perturbation method, the solution to known deterministic inputs is expressed as a power series in  $\epsilon$ , a perturbation parameter which in our problem is only a mathematical artifice. We have an exact solution when  $\epsilon = 0$  but the solution we seek is obtained by letting  $\epsilon = 1$ . The response of the time variable system is then calculated by repeatedly solving the associated constant parameter system for known inputs and using the principle of superposition. Finally, the computation of the response correlation function comprises single quadrature for stationary inputs and double quadrature for non stationary inputs. (See equations 3.8 and 2.6)

A direct Fourier inversion of the correlation function to obtain the spectral description of the response, equation 2.4, involves computationally inconvenient quadrature operations which can be avoided if the time variable parameters and the input modulating function are periodic as in the case of a flapping blade. In order to make use of the periodicity of the system parameters, we express the stochastic response  $\beta(t)$  as a power series in  $\epsilon$  and then obtain the double frequency spectrum  $S_{\beta}(\omega_1, \omega_2)$  according to equation 2.4. This latter perturbation scheme is analogous to the one employed for non linear stochastic problems, Ref. 2, page 272, except in the present blade flapping problem the excitation is a special kind of a non stationary process for which the spectral density function comprises series of line masses, equation 2.21. The response spectrum same as the input spectrum also contains line masses. Therefore, for the physically realizable time

averaged response spectrum one needs to consider only the diagonal terms of the response spectrum.

#### 4.3.1 Computation of Response Correlation Function

Consider a time variable parameter linear system typified by the equation

$$F(\beta) = \beta^n + \sum_{j=1}^n \{c_j + d_j \sin(\omega_0 t + \theta_j)\} \beta^{n-j} \quad 4.3$$

where

$$\beta^n = \frac{d^n \beta}{dt^n}$$

With the forcing function  $f(t)$  for which the spectral density function  $S_f(\omega_1, \omega_2)$  or the correlation function  $R_f(t_1, t_2)$  is known, equation 4.3 takes the form

$$F(\beta) = f(t) \quad 4.4$$

Now introduce two more linear operators

$$L(\beta) = \beta^n + \sum_{j=1}^n c_j \beta^{n-j} \quad 4.5$$

and

$$N(\beta) = L(\beta) - F(\beta) \quad 4.6$$

If

$$|c_j| \gg |d_j \sin(\omega_0 t + \theta_j)| \quad 4.7$$

it is possible to introduce a perturbation parameter  $\epsilon$  such that (10, 11)

$$\beta(t) = \beta_0(t) + \epsilon \beta_1(t) + \epsilon^2 \beta_2(t) + \dots \quad 4.8$$

Instead of solving equation 4.3, we now seek a solution to the problem

$$L(\beta) = f(t) + \epsilon N(\beta) \quad 4.9$$

for  $\epsilon = 1$ .

Substituting the power series expansion, equation 4.8, in equation 4.9 and equating the coefficients having the same powers of  $\epsilon$  one gets the following infinite system of equations:

$$L(\beta_0) = f(t) \quad 4.10$$

$$L(\beta_1) = - \left[ \sum_{j=1}^n d_j \sin(\omega_0 t + \theta_j) \beta_0^{n-j} \right] \quad 4.11$$

$$L(\beta_{k+1}) = - \left[ \sum_{j=1}^n d_j \sin(\omega_0 t + \theta_j) \beta_k^{n-j} \right]. \quad 4.12$$

In equation 4.4, when the random input  $f(t)$  is replaced by a deterministic harmonic forcing function  $e^{i\omega t}$ , the response  $y(\omega, t)$  can also be expressed as

$$y(\omega, t) = y_0(\omega, t) + \epsilon y_1(\omega, t) + \epsilon^2 y_2(\omega, t) + \dots$$

The solution of equation 4.10 with  $f(t)$  replaced by  $e^{i\omega t}$  gives

$$y_0(\omega, t) = H(\omega) e^{i\omega t} + \sum_{j=1}^n B_j(\omega) e^{i\omega_j t} \quad 4.13$$

where  $\omega_j$  are the roots of the equation

$$(i\lambda)^n + \sum_{j=1}^n c_j (i\lambda)^{n-j} = 0$$

and the constants  $B_j(\omega)$  are to be evaluated by satisfying the  $n$  zero initial conditions of the system. The first part of the response in equation 4.13 refers to the steady state

solution and the term under the summation sign refers to transient solutions which can be neglected for stable systems assuming one is only interested in the steady state. If transients are of interest, they can always be superimposed to the stochastic solution. Henceforth, we will consider only steady state solutions.

When  $y_0(\omega, t)$  and its  $n-1$  derivatives are substituted in equation 4.11 and noting that the right hand side deterministic input is periodic, it is possible to solve for  $y_1(\omega, t)$  in closed form. Similarly,  $y_2(\omega, t)$ ,  $y_3(\omega, t)$ ...etc. have to be solved if correction terms of order more than one are needed. As the system is linear one can set

$$y(\omega, t) \approx y_0(\omega, t) + y_1(\omega, t) + \dots \text{etc.} \quad 4.14$$

The correlating function  $R_\beta(t_1, t_2)$  is then obtained from the relation 3.8.  $y_c(\omega, t)$  and  $y_s(\omega, t)$  in equation 3.20, respectively correspond to inputs  $\cos \omega t$  and  $\sin \omega t$  or the real and imaginary parts of  $y(\omega, t)$  in equation 4.14.

In the present rigid blade flapping problem the forcing function in equation 4.4 is a separable non stationary process of the type of equation 2.16, where the input modulating function is periodic. Therefore, the spectral density function of the input is given by equation 2.21. In the computation of the response correlation function, the double integral in equation 3.8 reduces to a series of single integrals depending upon the number of Fourier terms in the input modulating function. Another approach which is better suited for a specific problem involving no parametric study of the input modulating function and especially when it is not periodic is to compute  $y_A(\omega, t)$  instead of  $y(\omega, t)$  and then use relation 3.26 to compute  $R_\beta(t_1, t_2)$ . The perturbation scheme to compute  $y_A(\omega, t)$  is exactly similar to the one described earlier. The only difference is that the forcing function now is a product of the input modulating function and  $e^{i\omega t}$ .

For any input modulating function  $A(t)$ ,  $y_{A0}(\omega, t)$ ,  $y_{A1}(\omega, t) \dots y_{Ak+1}(\omega, t)$ , etc. can be obtained from equations

$$y_{A0}(\omega, t) = \int_{-\infty}^{\infty} h(\tau) A(t - \tau) e^{i\omega(t - \tau)} d\tau \quad 4.15$$

$$y_{A1}(\omega, t) = \int_{-\infty}^{\infty} h(\tau) F_0(t - \tau) d\tau \quad 4.16$$

$$y_{Ak+1}(\omega, t) = \int_{-\infty}^{\infty} h(\tau) F_k(t - \tau) d\tau, \text{ etc.} \quad 4.17$$

where  $F_0(t), \dots, F_k(t)$  represent the right hand side deterministic functions in equations 4.11 and 4.12 after replacing the stochastic responses  $\beta_0(t), \dots, \beta_k(t)$  by the deterministic responses  $y_{A0}(\omega, t), \dots, y_{Ak}(\omega, t)$ , etc. When the input modulating function is periodic, convolution integrals in equations 4.15, 4.16 and 4.17 can be evaluated in closed form. As the system is linear

$$y_A(\omega, t) \simeq y_{A0}(\omega, t) + y_{A1}(\omega, t) + \dots, \text{ etc.} \quad 4.18$$

We note in passing that when the input is a stationary process,  $y(\omega, t)$  and  $y_A(\omega, t)$  are identical.

#### 4.3.2 Response Spectral Density Function

Taking the Fourier transform of equation 4.8 and then using relation 2.4, one gets

$$S_{\beta}(\omega_1, \omega_2) = S_{\beta_0}(\omega_1, \omega_2) + \epsilon [S_{\beta_0\beta_1}(\omega_1, \omega_2) + S_{\beta_1\beta_0}(\omega_1, \omega_2)] + \epsilon^2 \dots \quad 4.19$$

We are interested only in  $S_{\beta}(\omega, \omega)$  which for the specific type of input considered in this report corresponds to the

physically realizable time averaged power spectrum  $\bar{S}_\beta(\omega)$ .  
By virtue of relation 2.15, 3.18 and 3.19,  $\bar{S}_\beta(\omega)$  can be expressed as

$$\begin{aligned}\bar{S}_\beta(\omega) &= S_\beta(\omega, \omega) \\ &= S_{\beta_0}(\omega, \omega) + 2\epsilon \text{Real}[S_{\beta_0\beta_1}(\omega, \omega)] + \epsilon^2 \dots\end{aligned}\quad 4.20$$

Fourier transform of equation 4.10 at frequency  $\omega$  gives

$$B_0(\omega) = H(\omega) F(\omega)$$

Application of equation 2.4 gives the spectral density function in the form

$$S_{\beta_0}(\omega_1, \omega_2) = H^*(\omega_1) H(\omega_2) S_f(\omega_1, \omega_2) \quad 4.21$$

Similarly the spectral density functions for  $\beta_1(t), \dots$   
 $\beta_{k+1}(t)$ , etc. can be expressed as

$$\begin{aligned}S_{\beta_{k+1}}(\omega_1, \omega_2) &= 1/4 \left[ \sum_{j=1}^n \sum_{l=1}^n (-1)^{n-j} d_j d_l (1)^{2n-1-j} e^{-1(\theta_j - \theta_l)} \right. \\ &\quad \left. (\omega_1 - \Omega_0)^{n-j} (\omega_2 - \Omega_0)^{n-l} \right] S_{\beta_k}(\omega_1 - \Omega_0, \omega_2 - \Omega_0) \\ &- 1/4 \left[ \sum_{j=1}^n \sum_{l=1}^n (-1)^{n-j} d_j d_l (1)^{2n-1-j} e^{1(\theta_j + \theta_l)} \right. \\ &\quad \left. (\omega_1 + \Omega_0) (\omega_2 - \Omega_0)^{n-l} \right] S_{\beta_k}(\omega_1 + \Omega_0, \omega_2 - \Omega_0) \\ &- 1/4 \left[ \sum_{j=1}^n \sum_{l=1}^n (-1)^{n-j} d_j d_l (1)^{2n-1-j} e^{-1(\theta_j + \theta_l)} \right. \\ &\quad \left. (\omega_1 - \Omega_0)^{n-j} (\omega_2 + \Omega_0)^{n-l} \right] S_{\beta_k}(\omega_1 - \Omega_0, \omega_2 + \Omega_0)\end{aligned}$$

$$+1/4 \left[ \sum_{j=1}^n \sum_{l=1}^n (-1)^{n-j} d_j d_l (i)^{2n-1-j} e^{i(\theta_j + \theta_l)} \right. \\ \left. (\omega_1 + \Omega_0)^{n-j} (\omega_2 + \Omega_0)^{n-1} \right] S_{\beta_k}(\omega_1 + \Omega_0, \omega_2 + \Omega_0) \quad 4.22$$

Now, take the Fourier transform of equation 4.12 at frequency  $\omega_2$  to yield

$$\frac{1}{H(\omega_2)} \beta_{k+1}(\omega_2) = - \left[ \sum_{j=1}^n \frac{d_j}{2i} \left\{ (i)^{n-j} (\omega_2 - \Omega_0)^{n-j} \beta_k(\omega_2 - \Omega_0)^{n-j} e^{i\theta_j} \right. \right. \\ \left. \left. - (i)^{n-j} (\omega_2 + \Omega_0)^{n-j} \beta_k(\omega_2 + \Omega_0)^{n-j} e^{-i\theta_j} \right\} \right]$$

Multiplying both sides with  $\beta_k^*(\omega_1)$  and then taking the expectation, equation 2.4, one gets

$$S_{\beta_k \beta_{k+1}}(\omega_1, \omega_2) = H(\omega_2) \left[ 1/2 S_{\beta_k}(\omega_1, \omega_2 - \Omega_0) \sum_{j=1}^n e^{i\theta_j} d_j (i)^{n-j-1} \right. \\ \left. (\omega_2 - \Omega_0)^{n-j} - 1/2 S_{\beta_k}(\omega_1, \omega_2 + \Omega_0) \sum_{j=1}^n e^{-i\theta_j} d_j (i)^{n-j-1} (\omega_2 + \Omega_0)^{n-j} \right] \quad 4.23$$

With the help of relations 4.21, 4.22 and 4.23, the response spectral density function can thus be obtained to any desired order.



## 5. Numerical Examples

The equation studied by the perturbation and numerical methods reads

$$\ddot{\beta} + (c_1 + d_1 \sin \Omega_0 t) \dot{\beta} + (c_2 + d_2 \cos \Omega_0 t) \beta = A(t)\bar{\alpha}(t) \quad 5.1$$

Selecting a time unit for which the rotor angular velocity  $\Omega_0 = 1$  and substituting  $c_1 = \frac{\gamma}{8}$ ,  $d_1 = \frac{\gamma\mu}{6}$ ,  $c_2 = 2c_1$ ,  $d_2 = d_1$  and  $A(t) = c_1 + 2d_1 \sin t$ , an approximate blade flapping equation valid up to moderate advance ratios is obtained. (For details of equation 5.1 refer to Phase I Report, page 17). For practical rotors the non dimensional-inertia number  $\gamma$  varies from 2-10, therefore we have assumed a typical value of  $\gamma = 4$  in the numerical examples. Note also that the system parameters  $c_1$  and  $d_1$  which are linearly related to the advance ratio  $\mu$  also appear in  $A(t)$ . However, the system parameter  $d_1$  is varied from 0 to 1 only in the left hand side of equation 5.1 without changing the right hand side deterministic function  $A(t)$ . The computational scheme with different values of  $d_1$  is thus associated with two specific functions

$$A(t) = 0.5 + 0.4 \sin t$$

and

$$S_{\bar{\alpha}}(\omega) = \frac{2}{0.25 + \omega^2}$$

As the function  $A(t)$  which corresponds to the actual physical system at an advance ratio of 0.3, is not simultaneously changed along with the system parameter  $d_1$ , the computed values of the time variable mean square response  $R_{\beta}(t,t)$  and the time averaged response spectra  $\bar{S}_{\beta}(\omega)$  correspond to the actual blade flapping problem only for  $d_1 = 0.2$ . The intent of this report is not so much to carry out an extensive parametric study of  $\gamma$  and  $\mu$  for different input spectra which are of interest in the atmospheric turbulence study but to establish the range of validity of the

perturbation method with regard to  $\mu$  and then within this admissible range develop a computational scheme to compute the time variable mean square response and the time averaged spectra of the flapping oscillations. With the computer program written in Fortran IV language, it is possible to evaluate  $R_{\beta}(t,t)$  for any desired values of  $\gamma$  and  $\mu$  when the stochastic input  $\bar{\alpha}(t)$  has a known spectral density function  $S_{\bar{\alpha}}(\omega)$ .

Now, coming to the actual description of the computational scheme and presentation of computer results, it is convenient to describe them in three stages:

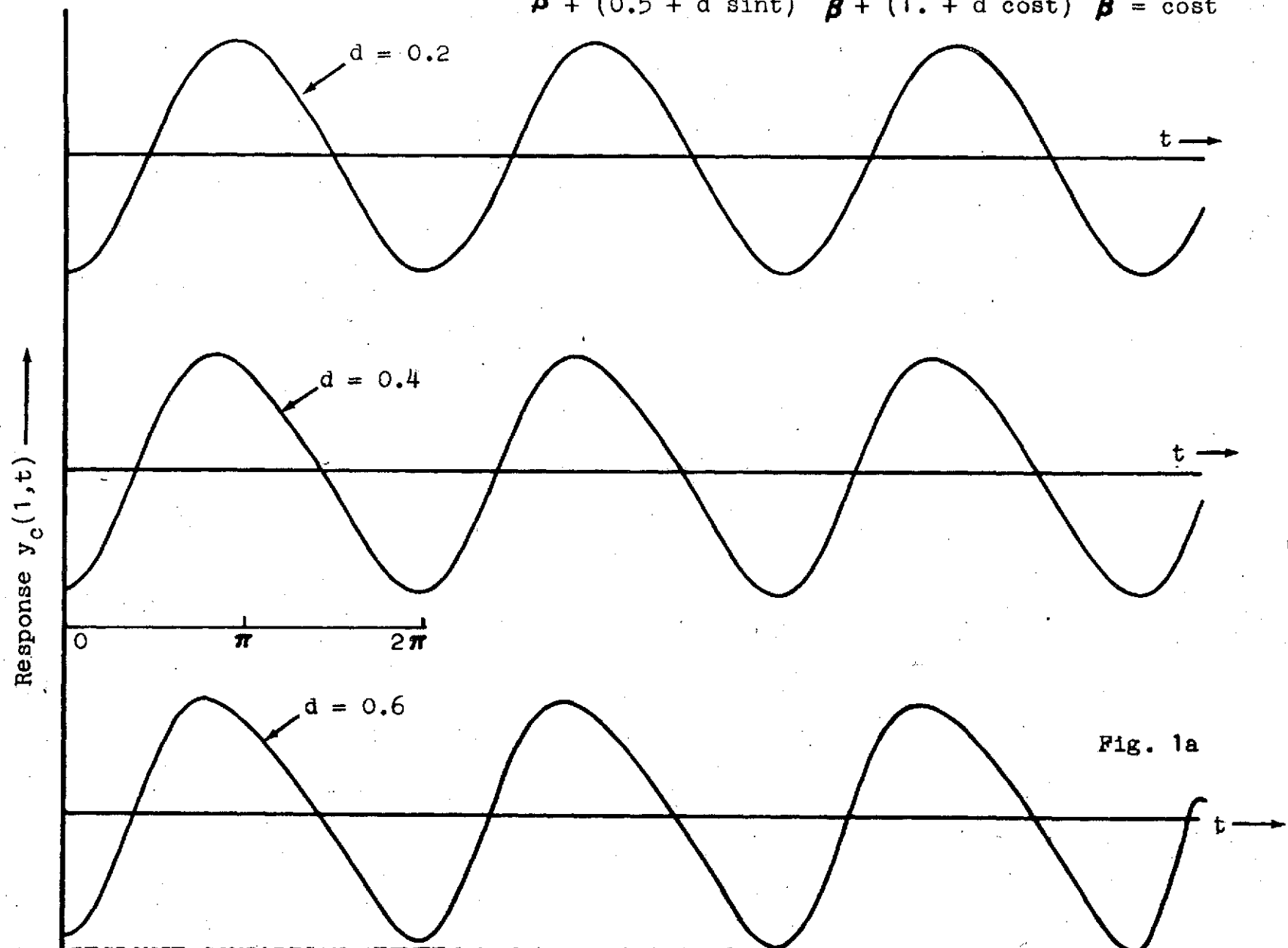
In the first stage, the range of validity of the perturbation method with respect to the advance ratio  $\mu$  has been established by comparing the perturbation system responses with that of the Runge-Kutta method results. The computer program, for any preassigned values of  $c_1$  and  $d_1$ , gives the system responses according to these two methods provided the input to the system is of the form  $(\frac{a_0}{2} + b_1 \sin t)e^{i\omega t}$ . Numerical results presented in Figures 1a to 1c refer to three typical frequency values of 0.5, 1 and 1.5 with constants  $a_0 = 2$  and  $b_1 = 0$ . A comparison of system responses, Figures 1a to 1c, indicates that up to  $d_1 = 0.7$  (or approximately  $\mu \leq 1$ ) the perturbation scheme should provide reasonably accurate results.

In the second stage, the time variable mean square response  $R_{\beta}(t,t)$  is evaluated according to equation 3.27, by integrating the product of the spectral density function  $S_{\bar{\alpha}}(\omega)$  and the square of the absolute value of the system response to the input  $(\frac{a_0}{2} + b_1 \sin t)e^{i\omega t}$ . The limits of integration in equation 3.2 have been truncated to -3 to +3. This finite frequency range of integration seems to be adequate for applied purposes because in the numerical examples discussed here the value of the integrand for  $|\omega| > 3$  is less than  $10^{-3}$ . The quadrature routine is based on Simpson's rule with a stepsize of 0.1. Figure 2 refers to two cases - stationary and non stationary inputs to a constant parameter system. The first case corresponds to the blade flapping equation at zero advance ratio and as expected, the mean

square response, dotted lines in Figure 2, are time invariant. The second case, though not directly relevant to the present blade flapping problem is an analytical model of considerable interest where the mean square response is time dependent due to non stationarity in the input. The strong time dependency of the mean square response, full lines in Figure 2, is due to the fact that the time invariant or the constant part of the input with  $\frac{a_0}{2} = 0.5$ , is not small compared to  $0.4 \sin t$ , the time variable part. For a spectral description of such a separable non stationary process refer to Section 2.3. Figure 3 shows the time dependency of the mean square response of time variable parameter systems subject to non stationary excitations. Here both the time variability of the system parameters and non stationarity of the input contribute toward the non stationarity of the response.

The third stage comprises the computation of the time averaged response spectra according to equations 4.21, 4.22, 4.23 and 4.20. Figure 4 summarizes these numerical results for different values of the system parameter  $d_1$ . For two extreme values of  $d_1$ ,  $d_1 = 0$  and  $d_1 = 0.8$ , Figure 5 shows the comparison between the perturbation values and the NASA conducted simulator results, and Figure 6 also refers to a similar comparison with  $d_1 = 0.2$ . Considering the discrepancy in the time averaged input power spectrum between the simulator study and exact analytical values, Figure 5, and also other types of errors due to finite filter band width, etc. inherent to simulator results, the perturbation values within the admissible range of the perturbation scheme mentioned earlier agree reasonably well with the simulator results, Figures 5 and 6.

$$\ddot{\beta} + (0.5 + d \sin t) \dot{\beta} + (1. + d \cos t) \beta = \cos t$$



FIGURES 1a-1c:

# RESPONSE COMPARISON BETWEEN RUNGA-KUTTA AND PERTURBATION METHOD VALUES

(For negligible difference between the Runge-Kutta and the perturbation method values the responses are shown by full lines only. Otherwise, the dotted lines represent the Runge-Kutta method values and the full lines perturbation method values.)

$$\ddot{\beta} + (0.5 + d \sin t) \dot{\beta} + (1. + d \cos t) \beta = \cos t$$

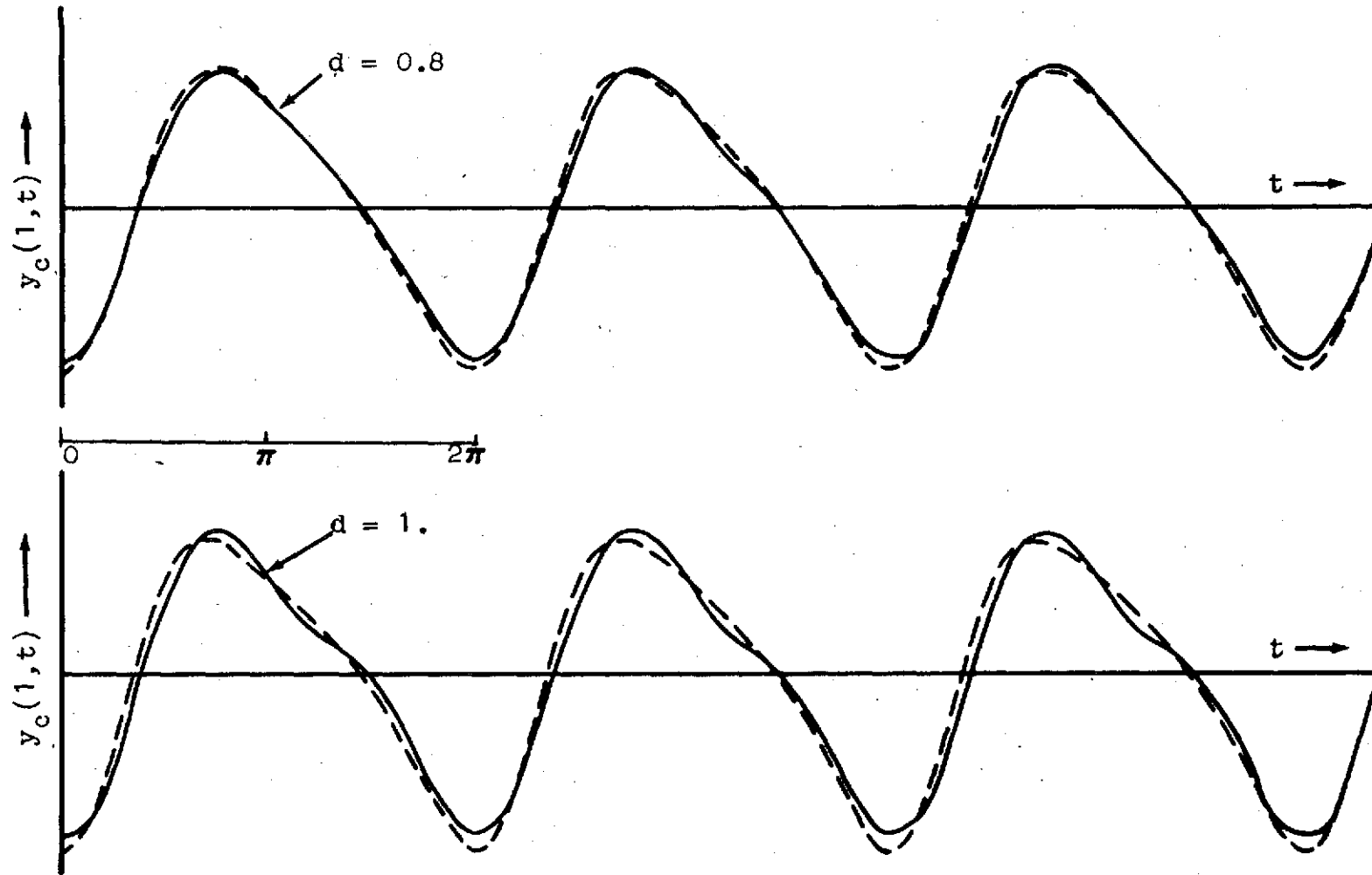


Fig. 1a continued

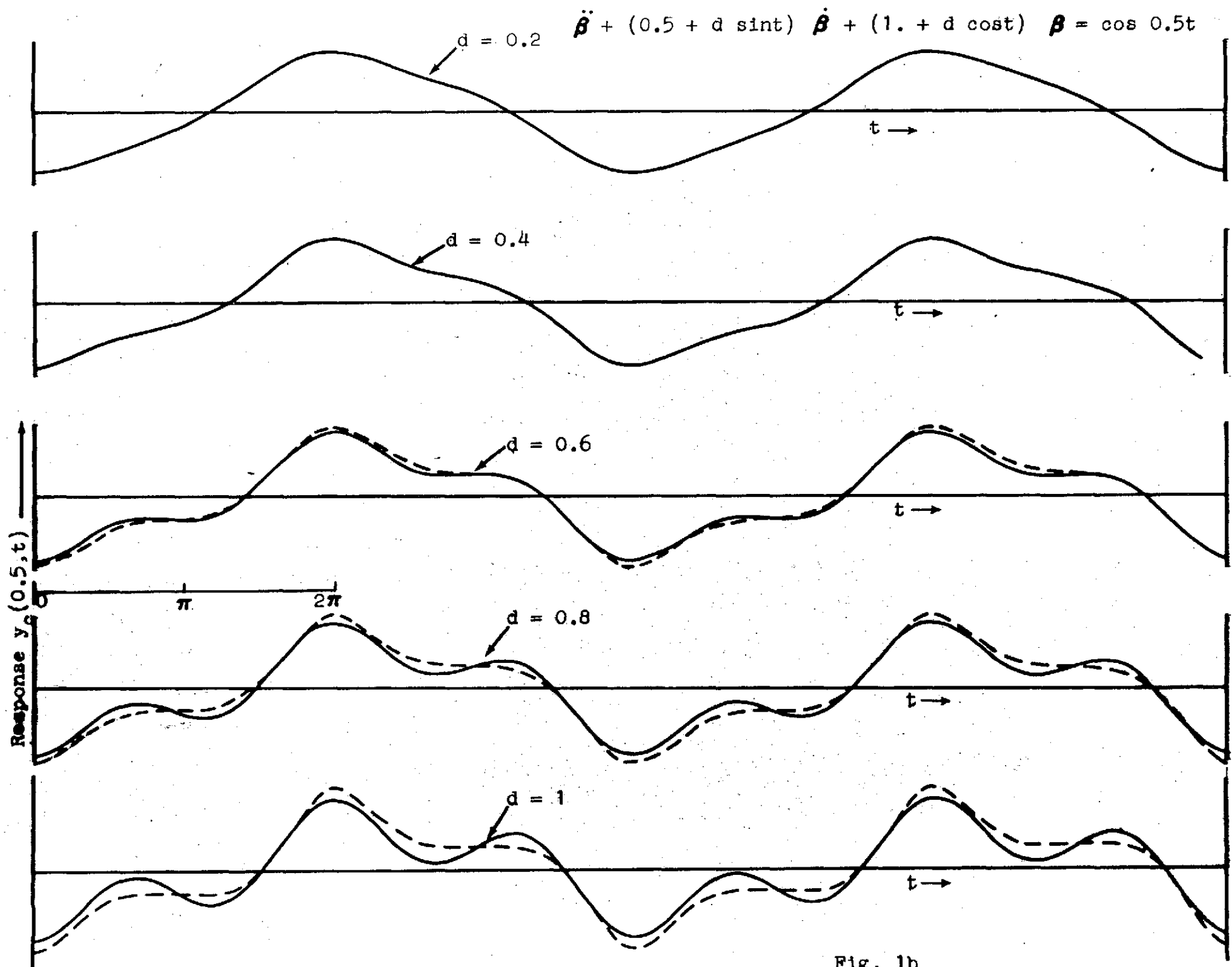


Fig. 1b

$$\ddot{\beta} + (0.5 + d \sin t) \dot{\beta} + (1. + d \cos t) \beta = \cos 1.5t$$

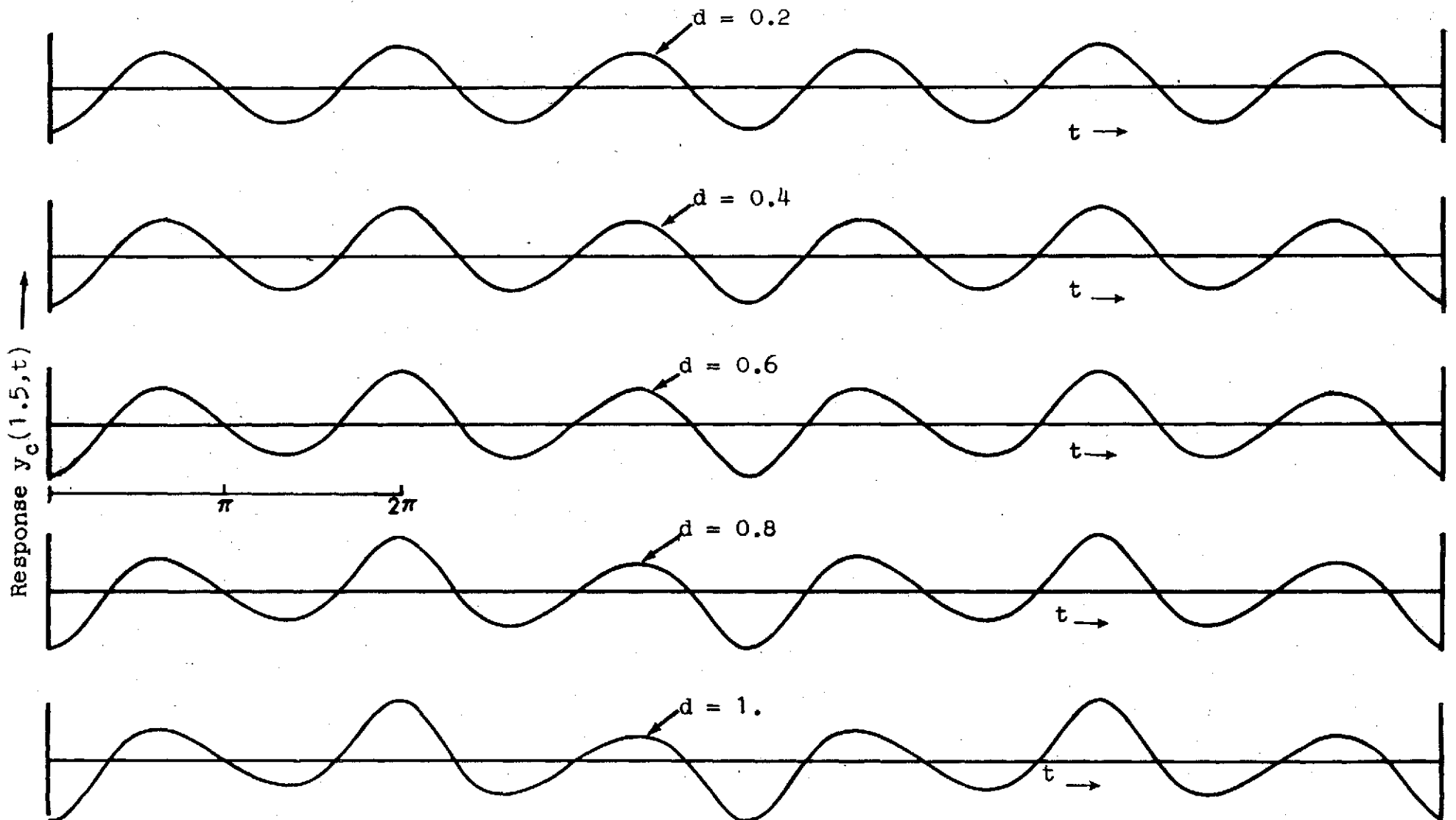


Fig. 1c

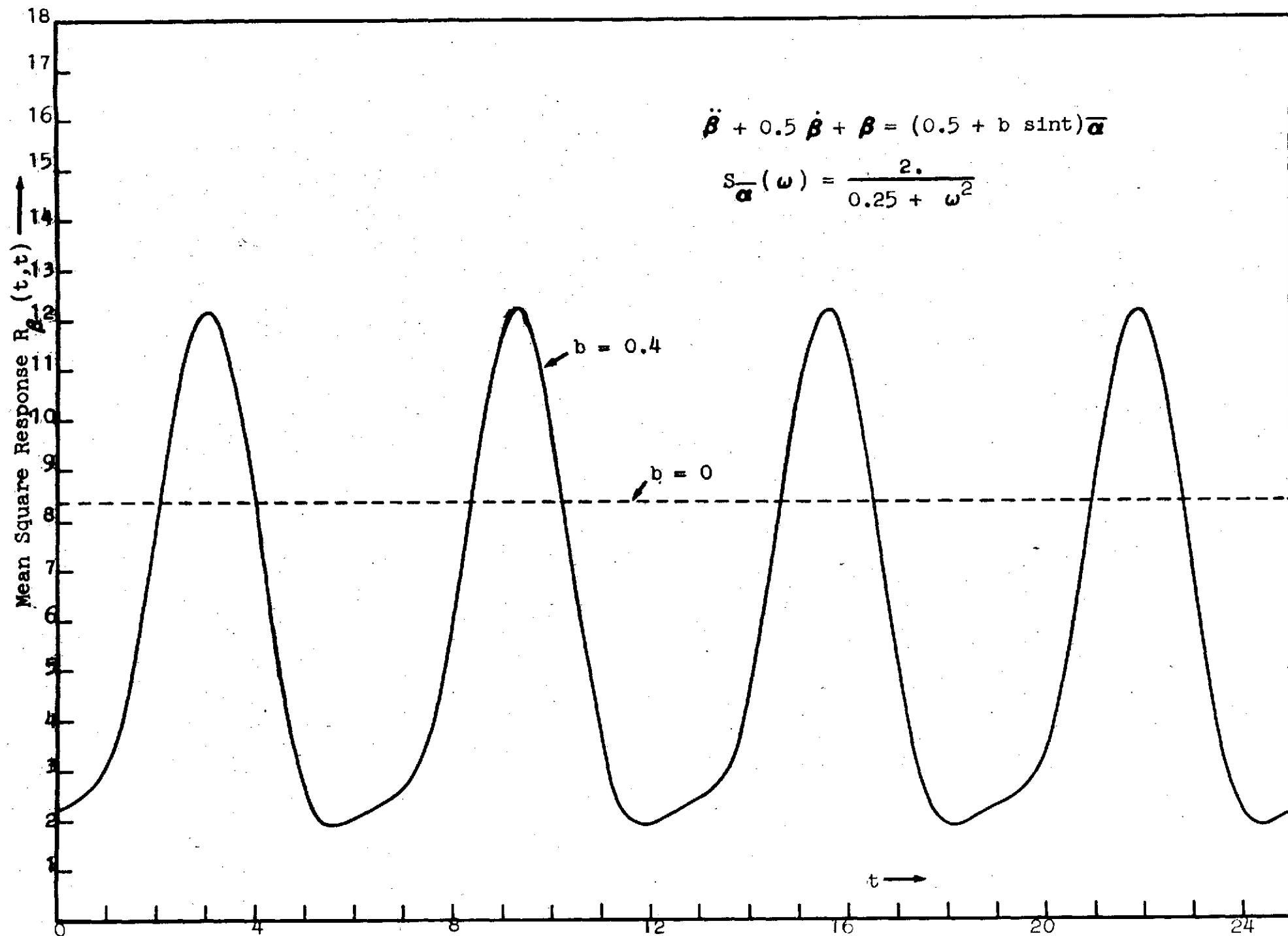


Figure 2: Mean Square Response of a Constant Parameter System Subject to Stationary and Non Stationary Excitations



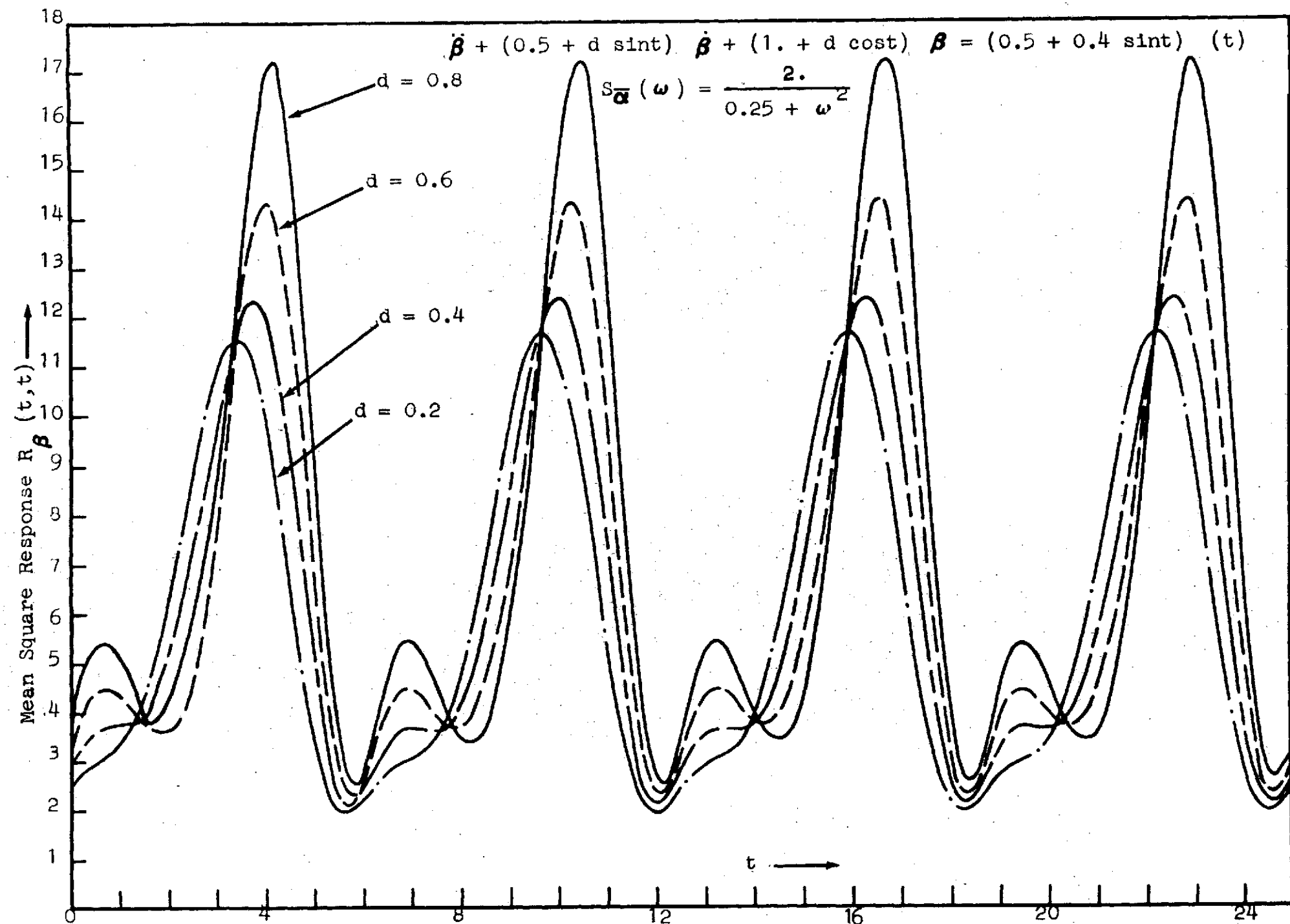


Figure 3: Study of the Effects of Time Varying Parameters on the Mean Square Response.  
 (The case  $d = 0.2$  refers to the blade flapping problem at an advance ratio  $\mu = 0.3$   
 and inertia number  $\gamma = 4$ )

$$\ddot{\beta} + (0.5 + a \sin t) \dot{\beta} + (1 + a \cos t) \beta = (0.5 + 0.4 \sin t) \bar{\alpha}$$

$$s_{\bar{\alpha}} = \frac{2}{0.25 + \omega^2}$$

INPUT:  
 --- exact

RESPONSE:  
 — a = 0.2  
 \* \* \* a = 0.3  
 ⊖ ⊖ ⊖ a = 0.5  
 ⊠ ⊠ ⊠ a = 0.8

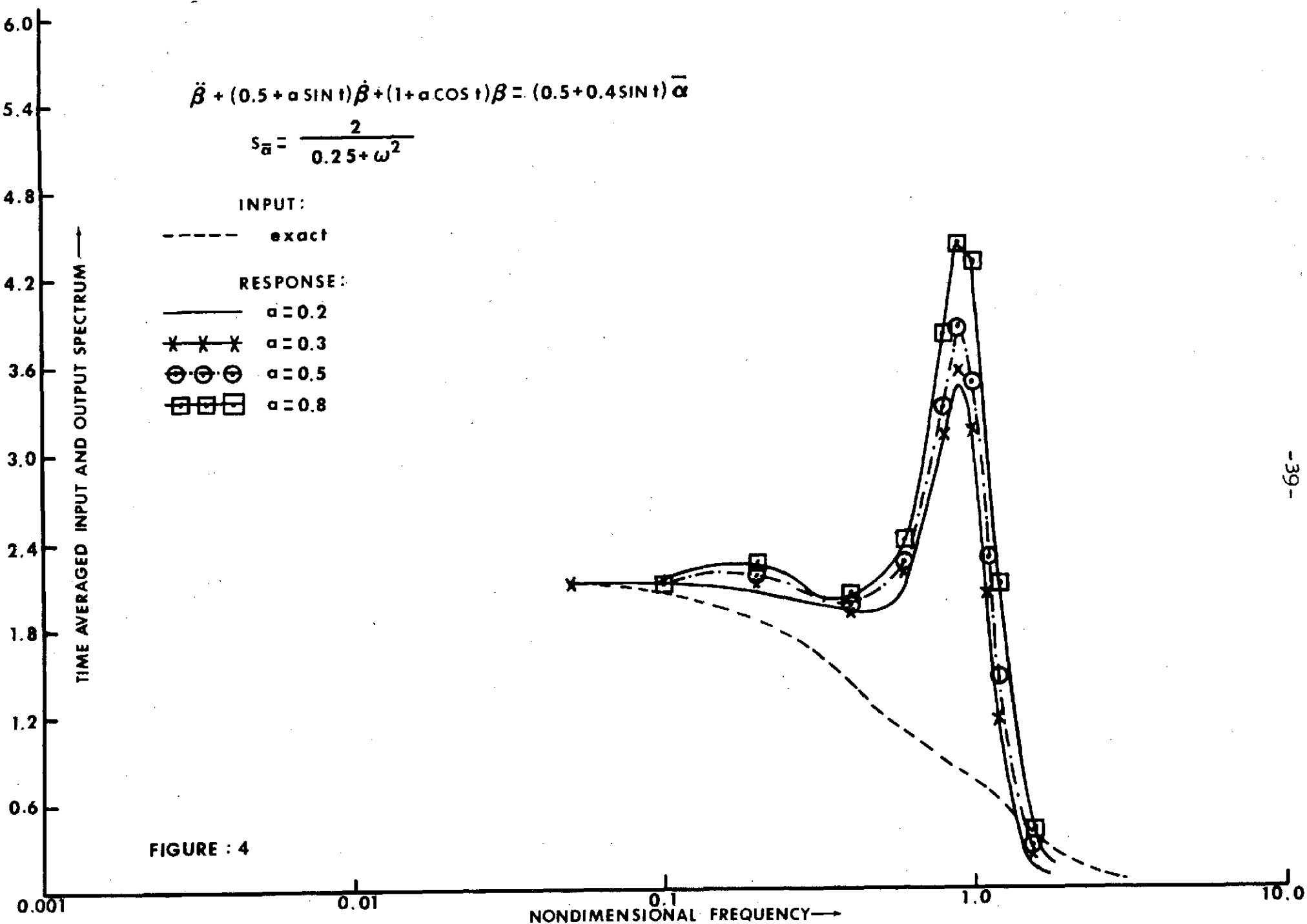
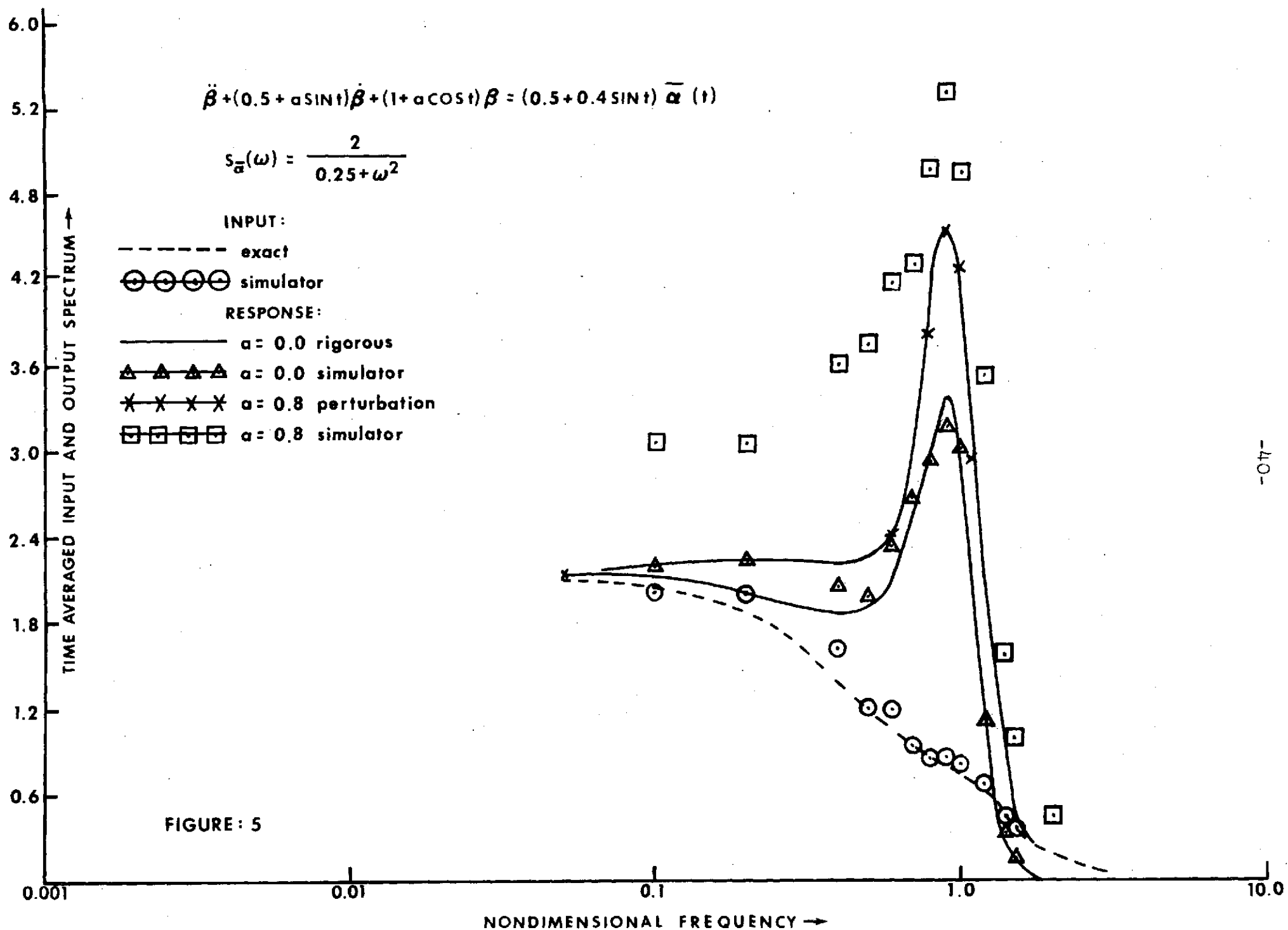


FIGURE : 4

STUDY OF THE EFFECTS OF TIME VARYING PARAMETERS ON THE TIME AVERAGED RESPONSE BY THE PERTURBATION METHOD.



NUMERICAL TIME AVERAGED RESPONSE COMPARISON BETWEEN SIMULATOR RESULTS AND PERTURBATION VALUES.

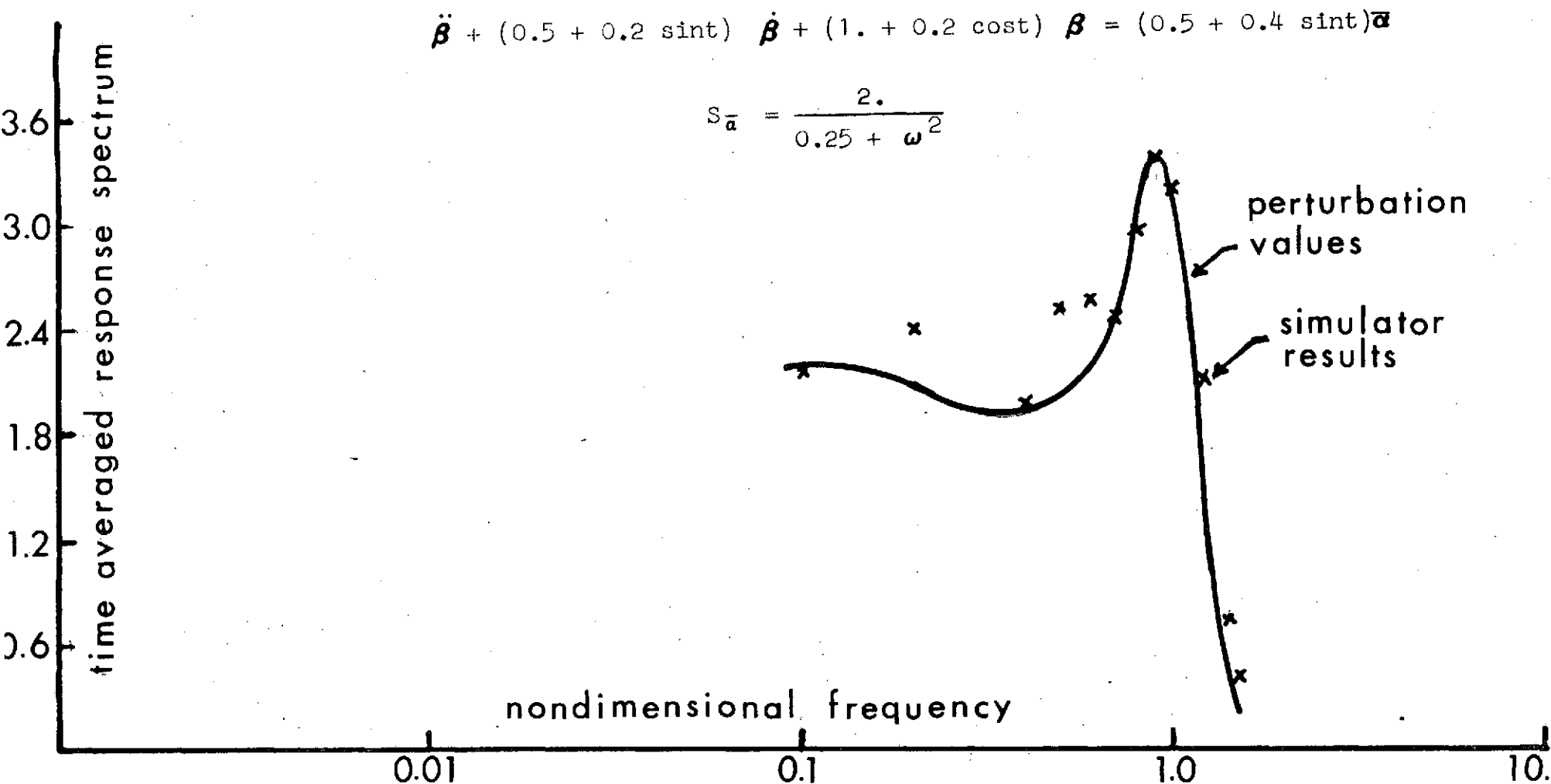


FIGURE 6: TIME AVERAGED SPECTRAL DENSITY OF FLAPPING OSCILLATIONS OF RIGID BLADES IN FORWARD HELICOPTER FLIGHT

## 6. Conclusions

Same as in Phase I Report it is assumed here that atmospheric turbulence produces an input stochastic loading of the lifting rotor blades in forward flight which is of the separable kind and consists of a stationary random process modulated by a periodic time function. Since submitting Phase I Report the problem of random blade flapping has been further studied and the following new results have been obtained:

- 6.1 The approximate method of solving the functional relation between input and output double frequency power spectral densities for the flapping rotor blade in forward flight, tentatively suggested in Phase I Report, has been checked against simulator results. While the method gives the correct order of magnitude effects of moderate advance ratio on the response power spectral density, it cannot be used as a quantitative estimate of these effects.
- 6.2 In the approximate method of 6.1 all off-diagonal terms of the double frequency input and output power spectral densities were neglected. A more elaborate approximation was tried, including the off-diagonal line masses. The resulting system of linear equations has a complex coefficient matrix which turned out to be ill conditioned and not directly suited for iterative techniques.
- 6.3 Solutions have been developed based on the system response to deterministic inputs. Such responses can be computed with standard numerical methods like the Runge-Kutta method. The deterministic input consists of a harmonic forcing function with or without being modulated by the right - hand side deterministic time function. If the time variability of the system parameters compared to the associated constant parameters is not too large, the deterministic response can be obtained with adequate accuracy also by the perturbation method. In either case the response autocorrelation function or the time variable mean square

response can be computed by a single integration over the applicable frequency range.

- 6.4 A perturbation method has been developed in two forms. According to the first form the deterministic responses are computed by adding to the solution for the associated constant parameter system corrections from the time varying parameters. The deterministic responses thus obtained are then inserted into the appropriate integrals over the applicable frequency range, representing autocorrelation function or time variable mean square.

According to the second form of the perturbation method the response power spectral density for the associated constant parameter system is first computed and then improved by adding the necessary corrective cross spectral terms. While the first form of the perturbation method is best suited for the computation of the time variable mean square response, the second form of the perturbation method lends itself particularly well to the computation of the time averaged response power spectral density.

- 6.5 The numerical examples presented herein have the main purpose to determine the range of applicability of the perturbation method and to evaluate for some typical assumed cases the stochastic structure of the blade flapping response. The results of this method are compared to the results of NASA conducted simulator studies. Also some typical response time histories obtained with the perturbation method are compared to those obtained with the more elaborate Runge-Kutta numerical integration method. On the basis of the numerical examples treated it can be concluded that the perturbation method of determining blade responses to stochastic inputs is approximately valid up to an advance ratio of one in combination with a Lock inertia number of 4.

So far the studies were primarily concerned with questions of methodology in treating time variable systems under non-stationary stochastic inputs of the separable type. The methods have been applied to a simplified approximate differential equation of blade flapping in forward flight. No effort was made to predict the stochastic input from a given atmospheric turbulence structure. Probably such a prediction will require empirical parameters to be obtained from model or flight tests. The extension of the studies to multi-degree of freedom representations of the blades will require data on cross correlation functions between the generalized stochastic loads, which also should be based on experiments. A rather straightforward extension of the present studies concerns a more accurate representation of the blade in flapping or flap-bending with more complex expressions for the time variable damping and stiffness terms. Such an extension together with the computation of typical random response data over a wide range of blade parameters and stochastic inputs is presently in work.

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